PMM U.S.S.R.,Vol.52,No.1,pp.126-129,1988
Printed in Great Britain
0021-8928/88 10.00+0.00

COMPARISON THEOREMS FOR NON-STATIONARY PRESSURE FILTRATION PROBLEMS*

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Comparison theorems are stablished for problems concerned with spatial pressure filtration under an elastic flow regime. In the theorems proposed here the nature of the change in the solution (in the head, the pressure, the rate of filtration and the flow rate) where there are certain changes in the boundary and initial conditions and the form of certain bounding surfaces is investigated within the framework of the accepted model. An example of the application of the theorems is given. The theorems obtained also hold for planar problems.

1. Formulation of the problem. The problem of spatial pressure filtration is considered. It is assumed that the liquid and the ground are weakly compressible and that the equation /1, 2/

$$\operatorname{div}(\varkappa \operatorname{grad} h) = \gamma\beta \partial h/\partial t \tag{1.1}$$

holds in the domain Ω in the case of a pressure head $h(x, y, z, t)$.

Here, β is the coefficient of elastic capacity, γ is the density of the liquid and $\varkappa = \varkappa(x, y, z)$ is the coefficient of filtration, which is a continuously differentiable function in Ω and does not vanish or become infinite in $\bar{\Omega}$.

It is assume that the region of flow is bounded by permeable surfaces S_k on which $h = H_k(t), k = 1, 2, \ldots, j$ (j is the number of surfaces with different pressure heads) and surfaces L_m on which $\partial h/\partial n = \sigma_m(t), m = 1, 2, \ldots, i$ (the internal normal). In particular, there may be impermeable surfaces among the L_m, if $\sigma_m(t) \equiv 0$ for certain values of m. It is assumed that each surface L_m only comes into contact with surfaces S_k and is a surface of the Lyapunov type while the surfaces S_k consist of a finite number of surfaces of the Lyapunov type. The flow region is bounded but may be multiply connected. The function $h(x, y, z, t)$ is assumed to be continuous in $\bar{\Omega}$ and to have continuous partial derivatives with respect to the variables x, y and z of the first order in $\bar{\Omega}/\bigcup S_k$, of the second order in Ω and a continuous first derivative with respect to t.

For the correct formulation of the boundary value problem it is necessary to specify the initial pressure head distribution

$$h(x, y, z, 0) = \varphi(x, y, z), \quad (x, y, z) \in \bar{\Omega}$$

where $\varphi(x, y, z)$ is a continuous function in $\bar{\Omega}$. Since a continuous solution of the problem is sought, the following compatibility conditions must be satisfied:

$$\varphi(x, y, z) = H_k(0), \quad (x, y, z) \in S_k.$$

Here and everywhere subsequently, $k = 1, 2, \ldots, j$; $m = 1, 2, \ldots, i$.

2. Comparison theorems. These theorems presuppose a comparison of the solutions of two problems which differ in some way. It is assumed that the solutions of both the initial and the modified problem exists and that they satisfy the conditions which have been enumerated in Sect.1. In the following treatment the difference in the values of any quantity in the case of the initial and modified solutions is denoted by the square brackets.

The forcing-in concept which is used in the theorems means that the points of the modified surface lie within the initial flow region. By replacing a certain surface by a surface of another type, we mean the replacement of the boundary conditions on a given part of the boundary of a region. The expression "the values of the pressure head increase", which is encountered in the formulations of the theorems, denotes that $[h(t)] > 0$. Other analogous expressions are to be understood in the same way.

Theorem 1. In the formulations of the problems under consideration, suppose only $\varphi(x, y, z)$ and $H_k(t)$ possibly are different and $[\varphi(x, y, z)] \geqslant 0$ in Ω, $[H_k(t)] \geqslant 0$ in a certain interval of time $[t_1, t_2]$ $(t_1 \geqslant 0)$, and $[H_k(t)] = 0$ at the remaining instants of time. Subject to these conditions, the solutions of the problems being compared are identically the same at a certain instant of time τ if and only if $[\varphi] \equiv 0$ in $\bar{\Omega}$ and $[H_k(t)] \equiv 0$ when $0 \leqslant t \leqslant \tau$. If, at the instant $\tau > 0$ the solutions are not identically the same, then, when the conditions of the theorem are satisfied:

*Prikl.Matem.Mekhan.,52,1,160-163,1988

a) the values of the pressure head (and the pressure) increase in Ω on L_m by a limited amount:

$$0<[h]<\max\left\{\max_{t\in[t_1,t_2]}[H_k(t)],\ \max_{(x,y,z)\in\bar{\Omega}}[\varphi]\right\}$$

b) on those surfaces S_k, where $[H_k(\tau)]=0$, the values of the outgoing velocities increase while the incoming velocities fall off in magnitude or become outgoing velocities. Of course, Theorem 1 enables us to consider the changes in the initial distribution of the pressure head $\varphi(x,y,z)$ separately (or the changes in the values of $H_k(t)$) separately, if we put $[H_k(t)]\equiv 0$ for any $t\geqslant 0$ (or, correspondingly, $[\varphi]\equiv 0$ in Ω).

Theorem 2. The solutions of problems are considered whose formulations can only differ in $\varphi(x,y,z)$ and $\sigma_m(t)$ and $[\varphi(x,y,z)]\leqslant 0$ in $\bar{\Omega}$, $[\sigma_m(t)]\geqslant 0$ in a certain interval of time $[t_3,t_4]$ $(t_3\geqslant 0)$ while, at the remaining instant of time, $[\sigma_m(t)]=0$. Under these conditions, the solutions of the problems being compared with identically the same at a certain instant of time τ if an only if $[\varphi]\equiv 0$ in $\bar{\Omega}$ and $[\sigma_m(t)]\equiv 0$ when $0\leqslant t\leqslant\tau$. If, at an instant $\tau>0$, the solutions are not identically the same then, when the above-mentioned conditions are satisfied:

a) the values of the pressure head (or pressure) decline in Ω on L_m,

b) the values of the inflow velocities on the surfaces S_k increase while the outflow velocities fall off in magnitude or become outflow velocities.

Other conclusions can be drawn when there are additional data in the formulation of the problem. For example, let the value of the pressure head on one of the boundary surfaces S_k in a certain interval of time $[0,T]$ be the greatest (smallest) of the pressure head values in Ω. We shall call this surface the greatest (least) pressure head surface. The following theorem is formulated under the assumption that just one of the above-mentioned surfaces exist in the interval of time $[0,T]$. Certain sufficient conditions for satisfying this requirement will be presented in Sect.3.

Below the notation applying to the initial solution is labelled with a single asterisk while the notation referring to the modified solution is labelled with two asterisks.

Theorem 3. When the surface under the greatest (least) stress is forced in and also when a part of the surfaces S_k or L_m is replaced by the surface under the greatest (least) stress, the following holds under the assumption that $[\varphi(x,y,z)]\geqslant 0$ $([\varphi(x,y,z)]\leqslant 0)$ for any $t\in(0,T]$:

a) the values of the pressure head and pressure increase (decrease) in Ω^{**} on L_m

b) on the unmodified parts of the surfaces S_k, the values of the outflow (inflow) velocities increase while the inflow (outflow) velocities fall off in magnitude or become outflow (inflow) velocities. In particular, the values of the velocity on the unmodified part of the surface of greatest (least) pressure head decrease while the velocities and flow rate through the unmodified part of the surface with the smallest (greatest) pressure head, if there is one, increase.

Remark. Under the conditions of Theorems 1 and 2, when there are surfaces under the greatest or least stress, it is possible in such a manner to obtain additional affirmation regarding the change in the velocities on these surfaces and the flow rates (including the total flow rate) through these surfaces.

Proof of the theorems. The theorems are proved by an investigation of the difference $h_0=[h]$ in the intersection Ω_0 of the domains Ω^* and Ω^{**}. In Ω_0, the function h_0 satisfies Eq.(1.1) and possesses the same properties of continuity and differentiability as h.

Let us consider the proof of Theorem 1. According to the conditions of the theorem, the function h_0 satisfies the following boundary and initial conditions: $\partial h_0/\partial n=0$ on L_m for any $t\geqslant 0$; $h_0\geqslant 0$ when $t\in[t_1,t_2]$ and $h_0=0$ on S_k at the remaining instants of time; $h_0(x,y,z,0)=\varphi_0=[\varphi]\geqslant 0$ in $\bar{\Omega}_0\equiv\bar{\Omega}^*\equiv\bar{\Omega}^{**}$.

We will first show that $h_0\geqslant 0$ in $\bar{\Omega}_0$ at any $t\in I$, where $I=[0,\tau]$ and τ is a certain instant of time. In fact, if there are negative values of h_0 in the time interval I, the function h_0 has a negative minimum in $\bar{\Omega}_0$ for $t\in I$. It follows from the strong maximum principle for second-order parabolic equations /3/ that this minimum is only attained on the boundary of the domain Ω_0 at a certain $t\in[0,\tau]$ and at the initial instant in $\bar{\Omega}_0$. Since $h_0\geqslant 0$ on the surfaces S_k and $\varphi_0\geqslant 0$ in $\bar{\Omega}_0$, we arrive at the conclusion that the point where the minimum is attained must only be located on the surfaces L_m. All of the conditions of Theorem 1 regarding the sign of the oblique derivative (the SOD theorem) of /4/ are satisfied. According to this theorem $\partial h_0/\partial n>0$ at the point where the minimum is attained. This, however, constradicts the conditions of the theorem. This means that there cannot be negative minimum of h_0 and $h_0\geqslant 0$ in $\bar{\Omega}_0$ when $t\in I$.

Let us assume that $h_0(x,y,z,\tau)=0$. Then, from the strong maximum principle, it follows that, when account is taken of the condition $h_0\geqslant 0$ in Ω_0, $h_0\equiv 0$ in Ω_0 when $t\in(0,\tau]$. Whence, taking into account the continuity of the function h_0, we have $\varphi_0\equiv 0$ in $\bar{\Omega}_0$ and $[H_k(t)]\equiv 0$ when $t\in I$. It is known that, when the last two identities are satisfied $h_0(x,y,z,\tau)=0$ in $\bar{\Omega}_0$ /4/, that is, the solutions are identically the same as the instant τ if and only if $[\varphi]\equiv 0$ in $\bar{\Omega}_0$

and $[H_k(t)] \equiv 0$ when $t \in I$.

Now, at the instant τ, let the solutions not be identically the same. Then, by applying the strong maximum principle and the SOD theorem, it is possible to write the inequalities

$$\min\left\{\min_{(x,y,z)\in\bar{\Omega}_0} \varphi_0, \quad \min_{t\in[t_1,t_2]} [H_k(t)]\right\} < h_0(x, y, z, \tau) < \max\left\{\max_{(x,y,z)\in\bar{\Omega}_0} \varphi_0, \quad \max_{t\in[t_1,t_2]} [H_k(t)]\right\} \quad (x, y, z)\in\bar{\Omega}_0/\cup S_k$$

from which assertion a) of the theorem follows. On these S_k surfaces where $[H_k(\tau)] = 0$, a minimum of the function h_0 is attained and, according to the SOD theorem, $\partial h_0/\partial n > 0$ on these surfaces. At those points of these surfaces where the velocities for the initial problem are outgoing (the projection of the velocity on the normal v_n^* is negative), the condition $\partial h^*/\partial n > 0$ is satisfied, whence, when account is taken of the inequality $\partial h_0/\partial n > 0$, we have there that $\partial h^{**}/\partial n > 0$. This means that $v_n^{**} < 0$, that is, velocities which are outgoing for the initial problem remain outgoing in the case of the modified problem. The difference in the magnitude of the velocities at these points can then be written in the form $[v] = \varkappa \partial h_0/\partial n$ whence the assertion regarding the change in the outgoing velocities follows. By similar reasoning it can be shown that some of the ingoing velocities for the initial problem can become outgoing in the case of the modified problem and when this happens, the magnitudes of the velocities which have remained ingoing are reduced.

Theorem 1 is proved.

Theorems 2 and 3 are proved using analogous arguments and taking account of the following: in Theorem 2 we have $h_0 \equiv 0$ on the surfaces S_k, $\partial h_0/\partial n \geqslant 0$ when $t \in [t_3, t_4]$ and $\partial h_0/\partial n = 0$ on L_m at the remaining instants of time and $\varphi_0 \leqslant 0$ in $\bar{\Omega}_0$. In Theorem 3, there are the conditions $\varphi_0 \geqslant 0$ $(\varphi_0 \leqslant 0)$ in $\bar{\Omega}^{**}$ and $h_0 \geqslant 0$ $(h_0 \leqslant 0)$ when $t \in [0, T]$ on the forced-in part of the surface under the greatest (least) stress as well as on those parts of the surfaces S_k and L_m which have been replaced by the surface of greatest (least) stress.

3. Certain sufficient conditions for the applicability of Theorem 3. We now present certain sufficient conditions which ensure the existence of surfaces of greatest and least stress.

Proposition 1. Let the following conditions be satisfied:

1) for a certain value of the index p

$$\varphi(x, y, z) \leqslant H_p(0) \text{ in } \Omega, \text{ on } L_m.$$

2) $H_k(t) \leqslant H_p(t)$, $k \neq p$, $t \in [0, T]$, and, moreover, at any $t \in (0, T]$, the identity equality is impossible simultaneously for all values of k.

3) $\sigma_m(t) \geqslant 0$, $H_p'(t) \geqslant 0$, $t \in [0, T]$.

Then, at any instant of time $t \in [0, T]$ the surface S_p is the surface of greatest stress.

Proposition 2. At any instant of time $t \in [0, T]$, the surface S_p will be the surface of least stress if the conditions of Proposition 1 are satisfied with the opposite signs in the inequalities.

For the proof of Proposition 1 let us consider an arbitrary time interval $I = [0, \tau]$, where $\tau \leqslant T$ and introduce the function

$$M(\tau) = \max_{(x,y,z)\in\bar{\Omega},\ t\in I} h(x, y, z, t)$$

According to the strong principle of a maximum, the value $M(\tau)$ can only be attained on the boundary $\partial\Omega$ of the domain Ω and at the initial instant in $\bar{\Omega}$. According to conditions 1 and 2 and the compatibility condition

$$\max_{(x,y,z)\in\bar{\Omega}} \varphi(x, y, z) = H_p(0)$$

and, according to the SOD theorem and the first inequality from condition 3, the value $M(\tau)$ cannot be attained on the surfaces L_m at any $t \in (0, \tau]$. Then, by further taking account of condition 2 and the second inequality from condition 3, we may write

$$M(\tau) = \max_{(x,y,z)\in\partial\Omega,\ t\in I} h(x, y, z, t) = \max_{t\in I} H_p(t) = H_p(\tau)$$

Since the instant of time τ is arbitrary, this means that, for any $t \in [0, T]$, the value of the stress on the surface S_p is greatest in $\bar{\Omega}$.

Proposition 2 is proved in a similar manner.

Finally, conditions 3 in the propositions are not necessary and, for example, when conditions 1 and 2 and the first of the inequalities from condition 3 of **Proposition 1** are satisfied, the surface S_p may be the surface of greatest stress even when there is a decrease in $H_p(t)$ provided that this decrease is sufficiently slow.

4. An example of the application of the theorems. The theorems which have been proved can be used both for a theoretical investigation of the solutions of problems as well as with the aim of obtaining estimates of the required solution in terms of a known (or simpler) solution.

As an example, let us consider the planar-parallel flow of a liquid to an aperture S in a homogeneous layer bounded by an arbitrary recharge contour L. It is assumed that Eq.(1.1) holds for the pressure head $h(x, y, t)$ in Ω and $h = f(t)$ on S, $h = H > H_0$ on L, where $H_0 = \max_t f(t)$. Let us now introduce into the treatment a further two problems with the same boundary conditions but which are solved in circular regions, one of which contains the region Ω within it while the other lies wholly within Ω. For the initial pressure head distribution in each domain we shall take the solution which corresponds to the stationary problem when $h = f(0)$ on S. Then, from the theorems of Polozhii /5/ and Theorem 3 for the pressure head and the flow rate of the aperture Q at any $t \geqslant 0$, we have the estimates

$$h_R(x, y, t) < h(x, y, t) < h_r(x, y, t), \; Q_R < Q < Q_r$$

Here, the indices r and R refer to the solution of the problem in the internal and external circle respectively. In the case of a central aperture in a circular layer it is possible to obtain computational formulae for the functions h_r and h_R /6, 7/ from which the above-mentioned estimates follow.

The estimates can be significantly simplified if use is made of the solution of the problem on the inflow of liquid towards the aperture ignoring the compressibility of the liquid and the ground.

Let $f'(t) \leqslant 0$, $h^+(x, y, t)$ be the solution of the equation $\Delta h = 0$ with the above-mentioned boundary conditions and let $h^{++}(x, y)$ be the solution of this equation when $h = H_0$ on S. The estimates

$$h_R{}^+ < h < h_r^{++}, \; Q_R{}^+ < Q \qquad (4.1)$$

are then valid.

In fact, since the function $\partial h/\partial t$ also satisfies Eq.(1.1), from the principle of a maximum, taking into account the condition $f'(t) \leqslant 0$, we have that $\partial h/\partial t \leqslant 0$ in Ω. This means that the function $h_0 = h - h^+$ will satisfy Poisson's equation with a non-positive right-hand side at each instant of time whence, according to the maximum principle for elliptic equations /3/, $h_0 > 0$ in Ω. Further, by taking account of Theorem 3, we have a lower estimate for the pressure head and the flow rate. The upper estimate for the pressure head follows immediately from Theorems 1 and 3.

The solutions $h_R{}^+$ and $h_r{}^{++}$ have a simple form for any arrangement of the aperture, and the estimates (4.1) can therefore be readily written out in elementary form.

Similar estimates can also be obtained in the case when $f'(t) \geqslant 0$.

Other examples of the application of the theorems are also possible.

We note that the theorems which have been formulated are a generalization of the comparison theorems of Polozhii /5/ to the case of spatial filtration of a compressible liquid in compressible ground.

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Translated by E.L.S.